Graded rings and modules

Def: A graded ring is a ring R along w/ a direct sum decomposition $R = R_0 \oplus R_1 \oplus \dots$ s.t. $R_i R_j \subseteq R_{i+j}$.

$f \in R$ is homogeneous if $f \in R_i$, some i. $I \subseteq R$ is a homogeneous ideal if it's generated by homogeneous elements.

Ex: $R = k[x_1, \dots, x_n] = S_0 \oplus S_1 \oplus \dots$ where S_i is the vector space of homogeneous polynomials of degree d.

i.e. $S_0 = k$, S_1 gen. by $x_1, \dots, x_n$, S_2 gen. by $x_1x_2, x_1x_3, \dots$ as k-vector spaces.

Def: If $R = R_0 \oplus R_1 \oplus \dots$ is a graded ring, then a graded R-module is a module

$$M = \bigoplus_{-\infty}^{\infty} M_i$$

s.t. $R_i M_j \subseteq M_{i+j}$.

Ex: Let $R = k[x_1, \dots, x_n]$ w/ standard grading.

1.) $I \subseteq R$ a homogeneous ideal (gen. by hom. elts) then R/I is a graded module w/ grading determined by $R \to R/I$.

2.) Let $M = R$, with $M_{-1} = R_0$, $M_0 = R_1$, $M_i = R_{i+1}$.

i.e. $\deg x_i = 0$, $\deg 1 = -1$, etc.

We denote this $M = R(1)$ ("R twisted by 1").

(More generally, if M is a graded module $M(d) \cong M$ as modules, and $M(d)_e = M_{d+e}$.)

Context: If $I \subseteq R$ describes a variety X in projective space, then $\dim_k(I_d)$ is the dimension of the space of forms of degree d that vanish on X.

Def: Let M be a f.g. graded module over $R = k[x_1, \dots, x_n]$, w/ standard grading. The <u>Hilbert function of M</u> is

$$H_M(s) := \dim_k M_s.$$

(Since M is f.g., these dimensions are finite.)

Ex: If $M = R$, w/ standard grading,

$$H_M(s) = \begin{cases} 0 & \text{for } s < 0 \\ \binom{s+n-1}{n-1}, & s \geq 0 \end{cases}$$

Ex: $M = k[x,y]/(x^2, y^3)$

$$\begin{array}{ccccccccc} M = & M_0 & \oplus & M_1 & \oplus & M_2 & \oplus & M_3 \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & 1 & & x, y & & y^2, xy & & xy^2 \end{array}$$

So $H_M(s) = \begin{cases} 1 & \text{for } d = 0,3 \\ 2 & \text{for } d = 1,2 \\ 0 & \text{otherwise} \end{cases}$

It turns out as soon as d is sufficiently large, $H_M(s)$ behaves like a polynomial:

Thm (Hilbert): If M is a finitely generated graded module over $k[x_0, \dots, x_r]$, then $H_M(s)$ agrees w/ a polynomial of degree $\leq r$ for $s >> 0$.

This polynomial is $P_M(s)$, the <u>Hilbert polynomial of M</u>.

First we assert the following claim:

Claim: Let $H(s) \in \mathbb{Z}$ be defined for all natural numbers s. If $\tilde{H}(s) = H(s) - H(s-1)$ agrees w/ a polynomial w/ $\mathbb{Q}$ coefficients of deg $\leq n-1$ for $s \geq s_0$, then $H(s)$ agrees w/ one of deg $\leq n$ for $s \geq s_0$.

Pf: Elementary. See book.

Pf of thm: By induction on the number of variables.

Base case: 0 variables. Then M is a finite dimensional vector space, so $H_M(s) = 0$ for $s >> 0$. (say $\deg(0) = -1$).

Now say $r \geq 0$. Then consider the map $M \to M$ given by mult. by x_r. We want the map to preserve the grading though, so we write

$$0 \to K(-1) \to M(-1) \xrightarrow{x_r} M \to M/x_r M \to 0$$

w/ $K \subseteq M$ the kernel, so the above sequence is exact.

Looking at the degree s piece of each term, we get

$$H_{M/x_r M}(s) - H_M(s) + H_M(s-1) - H_K(s-1) = 0$$

Since x_r annihilates every elt of K and $M/x_r M$, they are both f.g. $k[x_0, \dots, x_{r-1}]$-modules, so by induction,

the outer two terms agree for $s >> 0$ w/ polynomials of deg $\leq r-1$, thus so does

$$H_M(s) - H_M(s-1),$$

and we're done by induction. $\square$

<u>Geometric context:</u>

$X \subseteq \mathbb{P}^r$ a projective algebraic variety

- degree "d" of the Hilbert poly is the dimension of X
- $d!$ (initial coeff) = the "degree" of X = # of points

intersecting a general plane of complementary dimension.

- Riemann-Roch Theorem computes the Hilbert poly (v. important in AG)

- In AG the "Chern classes" of the corresponding sheaf are encoded in the coeffs of the Hilbert polynomial.